Admissible initial growth for diffusion equations with weakly superlinear absorption

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Abstract We study the admissible growth at infinity of initial data of positive solutions of $\partial_t u - \Delta u + f(u) = 0$ in $\mathbb{R}_+ \times \mathbb{R}^N$ when $f(u)$ is a continuous function, *mildly* superlinear at infinity, the model case being $f(u) = u \ln^\alpha(1+u)$ with $1 < \alpha < 2$. We prove in particular that if the growth of the initial data at infinity is too strong, there is no more diffusion and the corresponding solution satisfies the ODE problem $\partial_t \phi + f(\phi) = 0$ on $\mathbb{R}_+$ with $\phi(0) = \infty$.

Contents

1 Introduction and formulation of the results

Let h be a continuous nondecreasing function defined on $\mathbb{R}_+$ and vanishing only at 0. It is well known that for any continuous and bounded function g belonging to $C_b^+(\mathbb{R}^N)$,

the cone of bounded nonnegative continuous functions on $\mathbb{R}^N$, there exists a unique weak solution $u := u_g \in C_b^+(\mathbb{R}_+ \times \mathbb{R}^N)$ of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} := \mathbb{R}_+ \times \mathbb{R}^N, \\ \lim_{t\to 0} u(t,.) = g \qquad & \text{locally uniformly in } \mathbb{R}^N. \end{aligned} \tag{1.1}$$

Furthermore, the solution u satisfies

$$0 \le u(x,t) \le \Phi_{\|g\|_{L^\infty}}(t) \qquad \forall (t,x) \in Q^\infty_{\mathbb{R}^N}, \tag{1.2}$$

where Φ_a is the solution of the following Cauchy problem:

$$\begin{aligned} \Phi_t + \Phi h(\Phi) = 0 \qquad & \text{on } \mathbb{R}_+, \\ \Phi(0) = a. & \end{aligned} \tag{1.3}$$

Since $h(0) = 0$ it follows easily that

$$\Phi_a(t) \geqslant 0 \quad \forall t > 0, \forall a \geqslant 0.$$

Moreover the family $\{\Phi_a(t)\}$ is monotonically increasing with respect to the parameter a, and the condition:

$$\int_c^\infty \frac{ds}{sh(s)} < \infty, \qquad c = \text{const} > 0, \tag{1.4}$$

is equivalent to the existence of a solution $\Phi_\infty(t)$ of equation in (1.3) such that $\Phi_\infty(t) < \infty$, $\forall t > 0$ and with infinite initial data, i. e.

$$\lim_{t\to 0} \Phi_\infty(t) = \infty.$$

Now we come to our main subject, to study problem (1.1) with an initial data $g(x)$ unbounded and tending to infinity at infinity. It is clear that the character of growth of $h(s)$ at infinity defines the class of initial functions g *of solvability* of problem under consideration. For example, if $h(s)$ is bounded, then the corresponding class of solvability is the Tikhonov class [7] $\{g : g(x) \leqslant c\exp(c_1|x|^2), c, c_1 = \text{const} < \infty\}$. When $h(s)$ tends to infinity at infinity, the class of admissible initial data is larger that the Tikhonov class. If $h(s)$ increases at infinity fast enough in the sense that condition (1.4) is satisfied, then problem (1.1) is solvable for any nonnegative continuous function g in the sense that there exists a *prospective minimal solution* $\underline{u}_g$ which is the limit when $n \to \infty$ of the solutions u_{g_n} of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ \lim_{t\to 0} u(t,.) = g\chi_{B_n} \qquad & \text{in } L^1(\mathbb{R}^N), \end{aligned} \tag{1.5}$$

where B_n denotes the open ball of radius n and χ_A is characteristic function of the set A, and there holds

$$0 \le \underline{u}_g \le \Phi_\infty. \tag{1.6}$$

But the main question is to know whether the prospective minimal solution is truly a solution with initial data $g(.)$. If it is the case we say that the prospective minimal solution is the minimal solution. Another assumption on h which plays a fundamental role in the study is the so-called Keller-Osserman condition (see [1], [6]),

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} < \infty \text{ for some } a > 0, \text{ where } H(t) = \int_0^s th(t)dt. \tag{1.7}$$

When this condition is satisfied, for any $R > 0$ there exist solutions of

$$\begin{aligned} -\Delta u + uh(u) = 0 \qquad \text{in } B_R \\ \lim_{|x|\to R} u(x) = \infty. \end{aligned} \tag{1.8}$$

This condition gives rise to a localization phenomenon thanks to which we prove an existence and uniqueness result of the solution with initial data g.

Theorem A *Assume $r \mapsto rh(r)$ is convex and satisfies* (1.7)*. Then for any $g \in C^+(\mathbb{R}^N)$, $\underline{u}_g$ is the minimal solution with initial data g. Furthermore it is the unique nonnegative solution of problem* (1.1).

In the class of functions $h(s)$ of the form $h(s) = s\ln^\alpha(1+s)$, condition (1.4) is equivalent to $\alpha > 1$, but condition (1.7) is equivalent to $\alpha > 2$. In general it is easy to show that condition (1.7) is stronger than condition (1.4) .

When $h(s)$ is a power function, the class of existence and uniqueness is much larger than the class of Th. A. A complete description of this existence and uniqueness class is based upon the notion of initial trace which has been thoroughly investigated by Marcus and Véron [3], [4] and Gkikas and Véron [2].

When

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} = \infty \qquad \forall a > 0, \tag{1.9}$$

uniqueness may not hold in the class of unbounded solutions. If, for any $b > 0$, V_b denotes the maximal solution of the following Cauchy problem

$$\begin{aligned} V_{rr} + \frac{N-1}{r}V_r - Vh(V) = 0 \qquad \text{on } (0, R_b) \\ V(0) = b \\ V_r(0) = 0, \end{aligned} \tag{1.10}$$

then $R_b = \infty$. Actually, multiplying (1.10) by V_r, we get easily

$$2^{-1}\frac{d}{dr}|V_r|^2 = \frac{d}{dr}H(V) - \frac{N-1}{r}|V_r|^2 \leqslant \frac{d}{dr}H(V).$$

Since $V_r(0) = 0$, we derive

$$V_r(r) \leqslant \sqrt{2}\sqrt{H(V(r))} \qquad \forall r > 0.$$

Integrating this last inequality we obtain the *a priori* estimate

$$V(R) = V_b(R) \leqslant \bar{V}_b(R) \qquad \forall R > 0,$$

where the function $\bar{V}_b(R)$ is defined by the identity:

$$F_b(\bar{V}_b(R)) = \sqrt{2}R \qquad \forall R > 0, \text{ where } F_b(v) := \int_b^v \frac{ds}{\sqrt{H(s)}},$$

(see e.g. [8] also). Moreover it is easy to see that for arbitrary $a > b > 0$, $V_a(r) \geqslant V_b(r)$ $\forall r > 0$. Actually, due to the monotonicity of h, there holds

$$V_{a_{rr}}(0) = \frac{1}{N} V_a(0) h(V_a(0)) = \frac{1}{N} a h(a) > \frac{1}{N} b h(b) = V_{b_{rr}}(0),$$

from (1.10). Since $V_{a_r}(0) = V_{b_r}(0) = 0$ it follows from this last inequality that the function $r \mapsto W(r) = V_a(r) - V_b(r)$ is increasing near $r = 0$; it remains increasing on whole $\mathbb{R}_+$, since, if we assume that there exists $r_0 > 0$ where W reaches a local maximum, then $W_r(r_0) = 0, W_{rr}(r_0) \leq 0$, but from equation (1.10) we have:

$$W_{rr}(r_0) = V_a(r_0) h(V_a(r_0)) - V_b(r_0) h(V_b(r_0)) > 0,$$

which is a contradiction. Furthermore, Nguyen Phuoc and Véron proved in [5] that if g satisfies

$$V_c(|x|) \leq g(x) \leq V_b(|x|) \qquad \forall x \in \mathbb{R}^N, \tag{1.11}$$

for some $b > c > 0$, then there exists at least two different solutions of (1.1) defined in $Q^\infty_{\mathbb{R}^N}$: the minimal one $\underline{u}_g$ which satisfies

$$\underline{u}_g(x,t) \leq \Phi_\infty(t) \qquad \forall (x,t) \in Q^\infty_{\mathbb{R}^N}, \tag{1.12}$$

and another one u_g such that

$$V_c(|x|) \leq u_g(x,t) \leq V_b(|x|) \qquad \forall (x,t) \in Q^\infty_{\mathbb{R}^N}. \tag{1.13}$$

It is not clear wether there exists a maximal solution or not. However, if g satisfies (1.11), then there exists a minimal solution $\underline{u}_{g,c,b}$ and a maximal one $\overline{u}_{g,c,b}$ in the class $\mathcal{E}_{c,b}(g)$ of solutions of problem (1.1), satisfying inequalities (1.13). These two solutions can be constructed by the following approximate scheme: we define the sequence $\{\underline{u}_n\}$ of solutions of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) = V_c(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) = g \qquad & \text{in } B_n; \end{aligned} \tag{1.14}$$

then it is easy to check using comparison principle that the sequence $\{\underline{u}_n\}$ is increasing and converges to $\underline{u}_{g,c,b}$. Similarly, the sequence $\{\overline{u}_n\}$ of solutions of the same equation in

$Q^\infty_{B_n}$ with the same initial data and boundary value $V_b(n)$ is decreasing and converges to $\overline{u}_{g,c,b}$.

In this paper we consider the case where the initial data g grows at infinity faster than any function V_b with arbitrary $b < \infty$. Our aim is to describe analogs of the "maximal" solution u_g from (1.13) and prospective minimal solution $\{\underline{u}_g\}$ from (1.5), (1.6). For any $a > 0$ we denote by $u := u_{a,n}$ the solution of

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) = V_a(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) = \min\{V_a, g\} \qquad & \text{in } B_n, \end{aligned} \tag{1.15}$$

Due to the comparison principle it is clear that $u_{a,n} \leqslant V_a$ in $Q^\infty_{B_n}$. The next result highlights a phenomenon of *instantaneous blow-up* of the maximal solution if the initial data grows too fast at infinity.

Theorem B *Assume $r \mapsto rh(r)$ is convex and satisfies* (1.4) *and* (1.9)*. If $g \in C^+(\mathbb{R}^N)$, satisfies*

$$\lim_{|x|\to\infty} \frac{g(x)}{V_a(|x|)} = \infty \qquad \forall a > 0, \tag{1.16}$$

then for arbitrary $m \in \mathbb{N}$ the sequence $\{u_{a,n}\}_{n>m}$ decreases and converges in $Q^\infty_{B_m}$ to a function u_a which is solution of (1.1) *with initial data* $\min\{V_a, g\}$*. Furthermore $u_a(t,x) \to \infty$ for any $(t,x) \in Q^\infty_{\mathbb{R}^N}$ as $a \to \infty$. Thus, the function identically equal to ∞ in $Q^\infty_{\mathbb{R}^N}$ can be considered as the "maximal" solution of problem* (1.1) *in the case of* (1.16).

Let us remark that in subsection 3.1 we find the asymptotic expression of the functions V_a for the model nonlinearities h, which makes the condition (1.16) more explicit.

A fundamental example of equations with nonlinearities satisfying (1.4) and (1.9) is provided by

$$\partial_t u - \Delta u + u \ln^\alpha(1+u) = 0 \qquad \text{in } Q^\infty_{\mathbb{R}^N} \tag{1.17}$$

with $1 < \alpha \leq 2$. With this specific type of nonlinearity we prove:

Theorem C *Assume $1 < \alpha < 2$ and $g \in C^+(\mathbb{R}^N)$, satisfies condition* (1.16)*, which due to Proposition 3.1 has now the following form*

$$\lim_{|x|\to\infty} g(x) \exp\left(-c_\alpha |x|^{\frac{2}{2-\alpha}}\right) = \infty, \quad c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}.$$

Then the prospective minimal solution $\underline{u}_g$ of (1.17) *with initial data g is Φ_∞.*

Notice that the two types of generalized approximative solutions of problem (1.1), obtained in Theorems B, C, "forget" the real initial condition from (1.1): in another words, they realize infinite initial jump.

2 The maximal solution

2.1 Proof of Theorem A

The fact that $\underline{u}_g$ is a solution of (1.1), and clearly the minimal one, is more or less standard, but we recall its proof for the sake of completeness since it contains the localization principle. For $m \in \mathbb{N}^*$ let v_m be the minimal solution of

$$\begin{aligned} -\Delta v + vh(v) &= 0 \qquad \text{in } \; B_m \\ \lim_{|x|\to m} v(x) &= \infty. \end{aligned} \tag{2.1}$$

Such a solution exists by [1] or [6] because (1.7) holds. It is nonnegative and radial as limit of the nonnegative radial functions $v_{m,k}$, $k \in \mathbb{N}^*$ which are the solutions of (2.1) with finite boundary data $v_{m,k} = k$ on ∂B_m. Moreover $v_{m,k}$, and thus v_m, is an increasing function of $|x|$. Clearly $v_m \geq 0$ and it is a stationary solution of (1.1) in $Q^\infty_{B_m}$. For $n \geq m$, let u_{g_n} be the solution of (1.5) and $\gamma_m = \max\{g(x) : |x| \leq m\}$. Then $v_m + \gamma_m$ is a super solution of (1.5) in $Q^\infty_{B_m}$ which dominates u_n on $\partial_\ell Q^\infty_{B_m} \cup \{0\} \times B_m$ Thus $v_m + \gamma_m \geq u_n$ in $Q^\infty_{B_m}$. The set of functions $\{u_n\}$ is bounded and uniformly continuous in $Q^T_{B_{m-1}}$, by standard regularity theory for parabolic equations, thus it converges uniformly therein to $\underline{u}_g$ and $\underline{u}_g\lfloor_{B_m\times\{0\}}= g$. This implies that $\underline{u}_g$ has g as initial data.

Assume now that u another solution with the same initial data g. We set $w = u - \underline{u}_g$. Since $r \mapsto rh(r)$ is convex and $u - \underline{u}_g$ is positive,

$$uh(u) \geq \underline{u}_g h(\underline{u}_g) + (u - \underline{u}_g)h(u - \underline{u}_g).$$

Therefore w is a subsolution of problem (1.1), and $w(t, x) \to 0$ as $t \to 0$, locally uniformly in $\mathbb{R}^N$. By the comparison principle

$$w(t, x) \leq v_n(x) \qquad \text{in } \; Q^\infty_{B_n},$$

where v_n satisfies (2.1) in B_n. Furthermore $n \mapsto v_n$ is decreasing with limit v_∞ as $n \to \infty$. The function v_∞ verifies

$$-\Delta v + vh(v) = 0 \qquad \text{in } \; \mathbb{R}^N.$$

Furthermore it is nonnegative, radial and nondecreasing with respect to $|x|$. In order to prove that $v = 0$, we return to v_n which satisfies

$$v_{n\,r} = r^{1-N}\int_0^r s^{N-1}v_n(s)h(v_n(s))ds \leq v_n(r)h(v_n(r))r^{1-N}\int_0^r s^{N-1}ds = \frac{r}{N}v_n(r)h(v_n(r)).$$

Thus

$$-v_{n\,rr} + v_n(r)h(v_n(r)) = \frac{N-1}{r}v_{n\,r} \leq \left(1 - \frac{1}{N}\right)v_n(r)h(v_n(r))$$

which implies

$$-v_{n\,rr} + \frac{1}{N}v_n(r)h(v_n(r)) \leq 0.$$

Integrating twice yields

$$\int_{v_n(r)}^{\infty} \frac{ds}{\sqrt{H(t)}} \geq \sqrt{\frac{2}{N}}(n-r), \tag{2.2}$$

where H has been defined in (1.7). If we had $v_\infty(r) > 0$ for some $r > 0$, it would imply

$$\infty > \int_{v_\infty(r)}^{\infty} \frac{ds}{\sqrt{2H(t)}} \geq \infty,$$

a contradiction. Thus $v_\infty(r) = 0$ and $w(t,x) = 0$. $\square$

2.2 Proof of Theorem B

We recall that (1.16) holds and that $u_{a,n}$ denotes the solution of (1.15). Since $V_a\lfloor_{Q^\infty_{B_n}}$ is the solution of the Cauchy-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u + uh(u) = 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n \\ u(t,x) = V_a(n) \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n \\ u(0,.) = V_a \qquad & \text{in } B_n, \end{aligned} \tag{2.3}$$

it is larger than $u_{a,n}$. Thus $u_{a,n+1}\lfloor_{\partial_\ell Q^\infty_{B_n}} \leq u_{a,n}\lfloor_{\partial_\ell Q^\infty_{B_n}} = V_a$. Since $u_{a,n}(0,.) = u_{a,n+1}\lfloor_{B_n}(0,.)$ it follows that $u_{a,n+1}\lfloor_{Q^\infty_{B_n}} \leq u_{a,n}$. Then $\{u_{a,n}\}$ is a decreasing sequence, and its limit u_a is a solution of (1.1), which the first claim. By the same argument, $u_{a,n} \leq u_{b,n+1}\lfloor_{Q^\infty_{B_n}}$ in $Q^\infty_{B_n}$ for $b > a$. Hence $u_a \leq u_b$. We introduce the sequence $\{r_a\} : r_a \to \infty$ as $a \to \infty$ defined by:

$$r_a = \inf\{r > 0 : g(x) \geqslant V_a(x) \quad \forall\, |x| \geqslant r\}, \tag{2.4}$$

and, for $n \geq r_a$, we set $w_{a,n} = V_a - u_{a,n}$. By convexity $w_{a,n}$ satisfies

$$\begin{aligned} \partial_t w_{a,n} - \Delta w_{a,n} + w_{a,n} h(w_{a,n}) \leq 0 \qquad & \text{in } Q^\infty_{B_n} := \mathbb{R}_+ \times B_n, \\ w_{a,n}(t,x) = 0 \qquad & \text{in } \partial_\ell Q^\infty_{B_n} := \mathbb{R}_+ \times \partial B_n, \\ w_{a,n}(0,x) = (V_a - g)_+ \qquad & \text{in } B_n. \end{aligned} \tag{2.5}$$

Therefore

$$w_{a,n}(t,x) < \Phi_\infty(t) \qquad \text{in } Q^\infty_{B_n}, \tag{2.6}$$

where Φ_∞ is defined in (1.3) with $a = \infty$. Actually,

$$\int_{\Phi_\infty(t)}^{\infty} \frac{ds}{sh(s)} = t. \tag{2.7}$$

Notice also that the sequence $\{w_{a,n}\}$ is increasing and it converges, as $n \to \infty$, to $w_a = V_a - u_a$, which is dominated by Φ_∞ Thus

$$u_a(t,x) \geq V_a(x) - \Phi_\infty(t) \geq a - \Phi_\infty(t) \qquad \text{in } Q^\infty_{\mathbb{R}^N}. \tag{2.8}$$

Letting $a \to \infty$ implies the claim. $\square$

3 The prospective minimal solution

In this section we consider equation (1.17) with $1 < \alpha < 2$.

3.1 The stationary problem

Proposition 3.1 *Assume* $1 < \alpha < 2$, $a > 0$ *and* V_a *is the solution of*

$$\begin{aligned} V_{rr} + \frac{N-1}{r} V_r - V \ln^{\alpha}(V+1) = 0 \qquad &\textit{in } \mathbb{R}_+ \\ V_r(0) = 0 \\ V(0) = a. \end{aligned} \tag{3.1}$$

Then

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \qquad \textit{as } r \to \infty, \tag{3.2}$$

where $c_\alpha = \left(\frac{2-\alpha}{2}\right)^{\frac{2}{2-\alpha}}$.

Proof. We write $W = \ln(V+1)$. Since V is increasing $W > 0, W_r \geq 0$ and

$$W_{rr} + W_r^2 + \frac{N-1}{r} W_r - (1 - e^{-W}) W^{\alpha} = 0 \qquad \text{in } \mathbb{R}_+. \tag{3.3}$$

Thus

$$W_{rr} + W_r^2 - (1 - e^{-W}) W^{\alpha} \leq 0.$$

If we set $\rho = W$ and $p(\rho) = W_r(r)$, then $\rho \in [a, \infty)$ and

$$pp' + p^2 - (1 - e^{-\rho}) \rho^{\alpha} \leq 0.$$

This is a linear differential inequality in the unknown p^2. Integrating yields

$$p^2(\rho) \leq 2e^{-2\rho} \int_a^{\rho} (e^{2s} - e^s) s^{\alpha} ds = \rho^{\alpha} + O(1). \tag{3.4}$$

Thus $W_r(r) \leq W^{\frac{\alpha}{2}}(r) + O(1)$ as $r \to \infty$ which implies

$$W(r) \leq c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \qquad \text{as } r \to \infty. \tag{3.5}$$

Due to (3.5) relation (3.4) yields also the following inequality

$$0 < W_r \leq c_\alpha^{\frac{\alpha}{2}} r^{\frac{\alpha}{2-\alpha}} (1 + o(1)).$$

Since $W(r) \to \infty$ as $r \to \infty$, it follows from (3.3) and (3.4) that for any $\epsilon > 0$ there exists $r_\epsilon > 0$ such that

$$W_{rr} + W_r^2 \geq (1 - \epsilon) W^{\alpha} \qquad \text{on } [r_\epsilon, \infty).$$

Integrating this ordinary differential inequality we get

$$W(r) \geq (1-\epsilon))c_\alpha r^{\frac{2}{2-\alpha}}(1+o(1)) \qquad \text{as } r \to \infty. \tag{3.6}$$

Since ϵ is arbitrary, we derive

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}}(1+o(1)) \qquad \text{as } r \to \infty. \tag{3.7}$$

From the above estimates, we can improve (3.6). Using (3.4) and (3.7) we deduce from (3.3):

$$pp' + p^2 = (1-e^{-\rho})\rho^\alpha - \frac{N-1}{r}W_r \geq (1-e^{-\rho})\rho^\alpha - c\rho^{\alpha-1},$$

from which it follows easily

$$p^2(\rho) \geq 2e^{-2\rho}\int_a^\rho e^{2s}(s^\alpha - c's^{\alpha-1})ds = \rho^\alpha + O(1), \tag{3.8}$$

by l'Hospital rule. Combined with (3.7) and (3.5), it implies

$$W(r) = c_\alpha r^{\frac{2}{2-\alpha}} + O(1) \qquad \text{as } r \to \infty. \tag{3.9}$$

Returning to V_a, we derive

$$V_a(r) = e^{c_\alpha r^{\frac{2}{2-\alpha}} + O(1)} \qquad \text{as } r \to \infty. \tag{3.10}$$

□

Remark. If $\alpha = 2$, the same method yields

$$V_a(r) = e^{e^r + O(1)} \qquad \text{as } r \to \infty. \tag{3.11}$$

3.2 Proof of Theorem C

We recall that the prospective minimal solution $\underline{u}_g$ is the limit, when $n \to \infty$ of the (increasing) sequence of solutions $\{u_{g_{\ell_n}}\}$ of

$$\begin{aligned} \partial_t u - \Delta u + u\ln^\alpha(u+1) &= 0 && \text{in } Q^\infty_{\mathbb{R}^N} \\ u(0,.) &= g\chi_{B_{\ell_n}} && \text{in } \mathbb{R}^N, \end{aligned} \tag{3.12}$$

where $\{\ell_n\}$ is any increasing sequence converging to ∞. Furthermore, if we replace g by its maximal radial minorant defined by $\tilde{g}(r) := \min_{|x|=r} g(x)$, it satisfies also (1.16). Because of (1.16) there exists a sequence $\{r_n\}$ tending to infinity such that

$$r_n = \inf\{r > 0 : \tilde{g}(s) \geq V_n(s) \ \ \forall s \geq r\},$$

then $\tilde{g}(r_n) = V_n(r_n)$.

Step 1: Estimate from below. Put

$$g_n(|x|) = \begin{cases} \min\{\tilde{g}(r_n), \tilde{g}(|x|)\} & \text{if } |x| < r_n \\ \tilde{g}(r_n) & \text{if } |x| \geq r_n. \end{cases}$$

Let $\underline{u}_{g_n}$ be the minimal solution of

$$\begin{aligned} \partial_t u - \Delta u + u \ln^\alpha(u+1) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ u(0,.) = g_n \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{3.13}$$

Then $\underline{u}_{g_n} \leq \Phi_\infty$. For any sequence $\{\ell_k\}$ converging to infinity and any fixed k, there exists n_k such that for $n \geq n_k$, there holds $g\chi_{B_{\ell_k}} \leq g_n$. Since the sequence $\{\underline{u}_{g_n}\}$ is increasing, its limit u_∞ is a solution of (1.3) in $Q_\infty^{\mathbb{R}^N}$ which is larger than $u_{g_{\ell_k}}$ for any ℓ_k, and therefore larger also than $\underline{u}_{\tilde{g}}$. However, since $g_n \leq \tilde{g}$, $u_\infty \leq \underline{u}_{\tilde{g}}$. This implies

$$u_\infty = \underline{u}_{\tilde{g}} \leqslant \Phi_\infty. \tag{3.14}$$

Next, since $\underline{u}_{g_n}(0,x) \leq g(r_n)$, it follows that $\underline{u}_{g_n}(t,x) \leq g(r_n)$. Let $\omega_n = \Phi_{g(r_n)}$, i.e. the solution of (1.3) with $a = g(r_n)$, then ω_n satisfies

$$\int_{\omega_n(t)}^{g(r_n)} \frac{ds}{sh(s)} = t,$$

and $\underline{u}_{g_n} \geq w_n$ where w_n is the minimal solution of

$$\begin{aligned} \partial_t w - \Delta w + w \ln^\alpha(\omega_n+1) = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ w(0,.) = g_n \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{3.15}$$

If we set $w_n(t,x) = e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds} z_n(t,x)$, then

$$\begin{aligned} \partial_t z_n - \Delta z_n = 0 \qquad & \text{in } Q^\infty_{\mathbb{R}^N} \\ z_n(0,.) = g_n \qquad & \text{in } \mathbb{R}^N. \end{aligned} \tag{3.16}$$

Since

$$z_n(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} g_n(y)dy,$$

we can write $w_n(t,x) = I_n(t,x) + J_n(t,x)$ where

$$I_n(t,x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}}{(4\pi t)^{\frac{N}{2}}} \int_{|y|\leq r_n} e^{-\frac{|x-y|^2}{4t}} g_n(y)dy, \tag{3.17}$$

and

$$J_n(t,x) = \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}} \int_{|y|>r_n} e^{-\frac{|x-y|^2}{4t}} dy. \tag{3.18}$$

Clearly

$$\begin{aligned} J_n(t,x) &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}}\int_{|y|>r_n+|x|} e^{-\frac{|y|^2}{4t}}dy \\ &\geq \frac{e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)}{(4\pi t)^{\frac{N}{2}}}\left(\int_{|z|>r_n+|x|} e^{-\frac{z^2}{4t}}dz\right)^N. \end{aligned} \tag{3.19}$$

This integral term can be estimated by introducing Gauss error function

$$\operatorname{ercf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz. \tag{3.20}$$

In dimension N, it implies easily

$$J_n(t,x) \geq e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds}\tilde{g}(r_n)\left(\operatorname{ercf}\left(\frac{r_n+|x|}{2\sqrt{t}}\right)\right)^N. \tag{3.21}$$

Since

$$\operatorname{ercf}(x) = \frac{e^{-x^2}}{x\sqrt{t}}(1+O(x^{-2})) \quad \text{as } x \to \infty,$$

we derive

$$J_n(t,x) \geq \frac{\tilde{g}(r_n)}{((r_n+|x|)^2 t)^{\frac{N}{2}}} e^{-\int_0^t \ln^\alpha(\omega_n(s)+1)ds - \frac{N(r_n+|x|)^2}{4t}}\left(1+O\left(\frac{t}{r_n^2}\right)\right). \tag{3.22}$$

We write $\tilde{g}(r) = \exp(\gamma(r)) - 1$ and set

$$A_n(t,x) = \gamma(r_n) - \int_0^t \ln^\alpha(\omega_n(s)+1)ds - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t.$$

In order to have an estimate on $\omega_n(s)$, we fix $t \leq 1$ and $\tilde{g}(r_n) \geq 1$. There exists $a_0 \geq 1$ such that

$$\min\left\{\frac{\omega_a(t)}{\omega_a(t)+1} : 0 \leq t \leq 1,\, a \geq a_0\right\} \geq \frac{1}{2}.$$

In such a range of a and t,

$$\begin{aligned} \omega' + \omega\ln^\alpha(\omega+1) &= \omega' + \frac{\omega}{\omega+1}(\omega+1)\ln^\alpha(\omega+1) \\ &\geq \omega' + \frac{1}{2}(\omega+1)\ln^\alpha(\omega+1), \end{aligned}$$

which yields

$$\ln^\alpha(\omega_n(s)+1) \leq \left(\frac{2\gamma^{\alpha-1}(r_n)}{2+(\alpha-1)s\gamma^{\alpha-1}(r_n)}\right)^{\frac{\alpha}{\alpha-1}}.$$

From this inequality, we derive

$$\begin{aligned} \int_0^t \ln^\alpha(\omega_n(s)+1)ds &\leq \int_0^t \left(\frac{2\gamma^{\alpha-1}(r_n)}{2+(\alpha-1)s\gamma^{\alpha-1}(r_n)}\right)^{\frac{\alpha}{\alpha-1}} ds \\ &\leq 2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\,d\tau. \end{aligned}$$

Therefore

$$A_n(t,x) \geq \gamma(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t \\ - 2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\,d\tau. \tag{3.23}$$

Step 2: The maximal admissible growth. We claim that

$$\liminf_{|x|\to\infty} |x|^{-\frac{2}{2-\alpha}}\ln\tilde{g}(|x|) > N^{\frac{1}{2-\alpha}} \Longrightarrow \lim_{n\to\infty}\underline{u}_{g_n}(t,x) = \Phi_\infty(t) \qquad \forall (t,x)\in Q^\infty_{\mathbb{R}^N}. \tag{3.24}$$

By replacing $\tau \mapsto (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}$ by its maximal value on $(0, t\gamma^{\alpha-1}(r_n))$,

$$2^{\frac{\alpha}{\alpha-1}}\gamma(r_n)\int_0^{t\gamma^{\alpha-1}(r_n)} (2+(\alpha-1)\tau)^{-\frac{\alpha}{\alpha-1}}\,d\tau \leq \gamma^\alpha(r_n)t.$$

Then

$$A_n(t,x) \geq \gamma(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t - \gamma^\alpha(r_n)t := B_n(t,x), \tag{3.25}$$

and

$$\partial_t B_n(t,x) = \frac{N(r_n+|x|)^2}{4t^2} - \frac{N}{2t} - \gamma^\alpha(r_n).$$

Thus

$$\partial_t B_n(t,x) = 0 \text{ and } t>0 \Longleftrightarrow t := t_n = \frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}}. \tag{3.26}$$

Therefore $A_n(t_n,x)$ is bounded from below by the maximum of $B_n(t,x)$ which is achieved for $t = t_n$ and

$$B_n(t_n,x) = \gamma(r_n) - N\ln(r_n+|x|) - \frac{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}}{4} \\ - \frac{N(r_n+|x|)^2\gamma^\alpha(r_n)}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}} - \frac{N}{2}\ln\left(\frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^\alpha(r_n)}}\right).$$

Since $r_n\to\infty$ as $n\to\infty$ it follows from last representation that

$$B_n(t_n,x) = r_n\gamma^{\frac{\alpha}{2}}(r_n)\left(\frac{\gamma^{1-\frac{\alpha}{2}}(r_n)}{r_n} - N^{\frac{1}{2}}(1+\nu_n(x))\right), \tag{3.27}$$

where $\nu_n(x)\to 0$ as $n\to\infty$ uniformly on any compact set in $\mathbb{R}^N$. Therefore if g satisfies

$$\liminf_{|x|\to\infty} |x|^{-\frac{2}{2-\alpha}}\ln\tilde{g}(|x|) > N^{\frac{1}{2-\alpha}}, \tag{3.28}$$

then there holds
$$J_n(t_n,x) \underset{n\to\infty}{\to} \infty \Longrightarrow \lim_{t_n\to 0} \underline{u}_{g_n}(t_n,x) = \infty, \tag{3.29}$$
uniformly on compact subsets of $\mathbb{R}^N$. We fix $m > 0$, denote by λ_m the first eigenvalue of $-\Delta$ in $H_0^1(B_m)$, with corresponding eigenfunction ϕ_m normalized by $\sup_{B_m}\phi_m = 1$ and set, for $\epsilon > 0$,
$$W_{m,\epsilon}(t,x) = e^{-(t+\epsilon)\lambda_m}\Phi_\infty(t+\epsilon)\phi_m(x) \qquad \forall (t,x) \in Q_\infty^{B_m}.$$
Then
$$\begin{aligned}\partial_t W_{m,\epsilon} - \Delta W_{m,\epsilon} + W_{m,\epsilon}\ln^\alpha(W_{m,\epsilon}+1) &= W_{m,\epsilon}\left(\ln^\alpha(W_{m,\epsilon}+1) - \ln^\alpha(\Phi_\infty(t+\epsilon)+1)\right)\\ &\le 0.\end{aligned}$$
Since $\underline{u}_{g_n}$ increases to the prospective minimal solution $\underline{u}_{\tilde{g}}$, it follows due to (3.29) that there exists n_ϵ such that
$$\underline{u}_{\tilde{g}}(t_{n_\epsilon},x) \ge \underline{u}_{g_{n_\epsilon}}(t_{n_\epsilon},x) \ge W_{m,\epsilon}(t_{n_\epsilon}+\epsilon,x) \qquad \forall x \in B_m.$$
Last inequality in virtue of comparison principle implies
$$\underline{u}_g(t,x) \ge W_{m,\epsilon}(t+\epsilon,x) \qquad \forall (t,x) \in Q_\infty^{B_m}, t \ge t_{n_\epsilon}.$$
Letting $\epsilon \to 0$ yields $\underline{u}_g \ge W_{m,0}$ in $Q_\infty^{B_m}$. Since $\lim_{m\to\infty}\phi_m(x) = 1$, uniformly on any compact subset of $\mathbb{R}^N$ and $\lim_{m\to\infty}\lambda_m = 0$ we derive $\underline{u}_{\tilde{g}} \geqslant \Phi_\infty$ and finally $\underline{u}_g \geqslant \Phi_\infty$. This inequality together with (3.14) leads to $\underline{u} = \Phi_\infty$. □

Remark. In the case $\alpha = 2$, there holds
$$\int_0^t \ln^2(\omega_n(s)+1)ds \le 4\gamma(r_n)\int_0^{t\gamma(r_n)}(2+\tau)^{-2}d\tau \le t\gamma(r_n). \tag{3.30}$$
Therefore (3.25) is replaced by
$$A_n(t,x) \ge \gamma(r_n) - t\gamma^2(r_n) - \frac{N(r_n+|x|)^2}{4t} - N\ln(r_n+|x|) - \frac{N}{2}\ln t := B_n(t,x). \tag{3.31}$$
A similarly, there exists $t_n > 0$ where $t \mapsto B_n(t,x)$ is maximum and in that case
$$\begin{aligned}B_n(t_n,x) = \gamma(r_n) - N\ln(r_n+|x|) - \frac{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}{4}\\ - \frac{N(r_n+|x|)^2\gamma^2(r_n)}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}} - \frac{N}{2}\ln\left(\frac{N(r_n+|x|)^2}{N+\sqrt{N^2+4N(r_n+|x|)^2\gamma^2(r_n)}}\right),\end{aligned}$$
which yields
$$B_n(t_n,x) = \gamma(r_n) - r_n\gamma(r_n)(N^{\frac{1}{2}} - \nu_n(x)), \tag{3.32}$$
where $\nu_n(x) \to 0$ as $n \to \infty$ uniformly on any compact set in $\mathbb{R}^N$. Thus $B_n(t_n,x) \to -\infty$ as $n \to \infty$. A similar type of computation shows that the expression $I_n(t,x)$ defined in (3.17) converges to 0, whatever is the sequence $\{r_n\}$ converging to ∞.

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